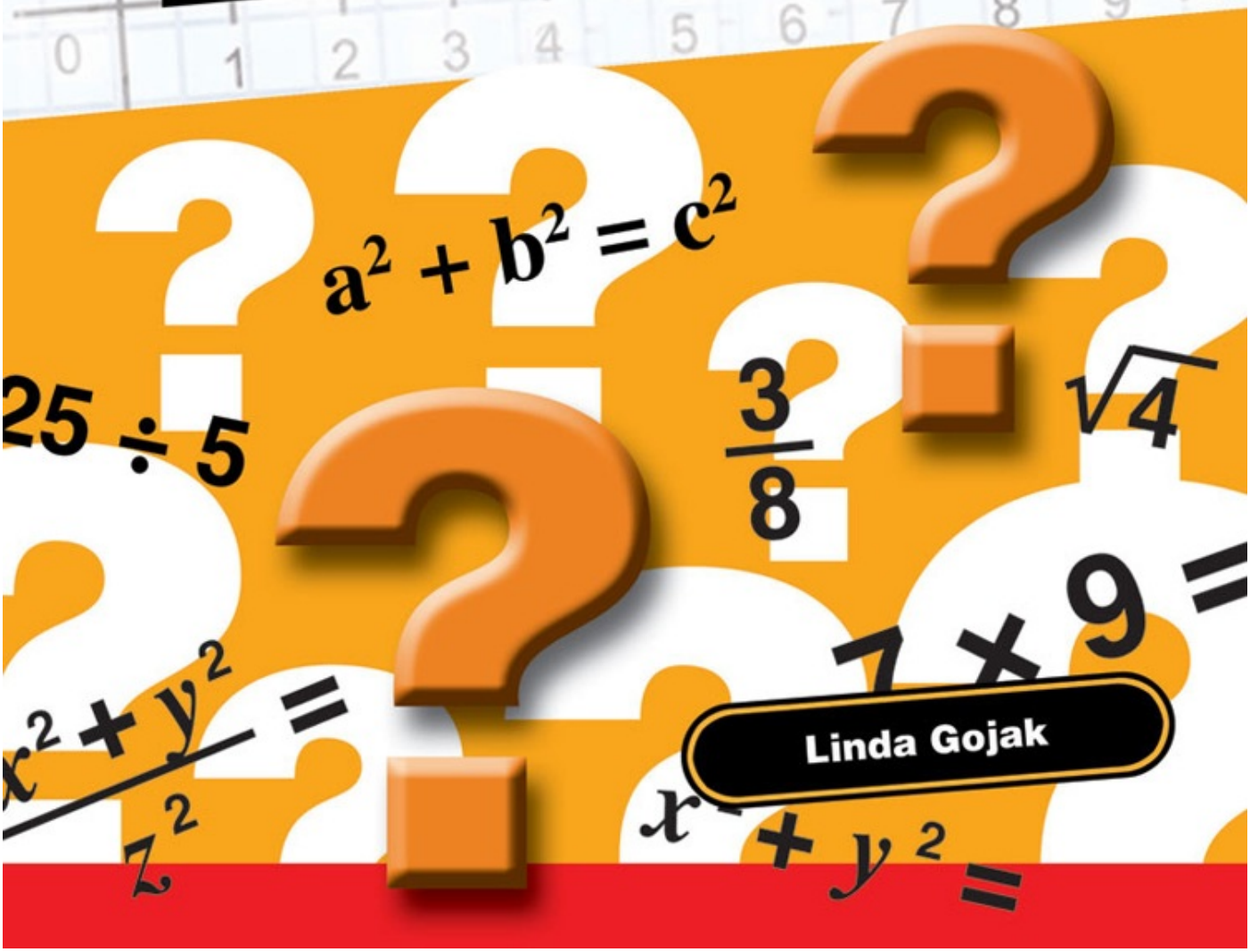
SHELL EDUCATION

WHAT'S YOUR MATH PROBLEM!?!

Getting to the Heart of Teaching Problem Solving



Linda Gojak

Publishing Credits

Dona Herweck Rice, *Editor-in-Chief;* Lee Aucoin, *Creative Director;* Don Tran, *Print Production Manager;* Sara Johnson, M.S.Ed., *Senior Editor;* Hillary Wolfe, *Editor;* Juan Chavolla, *Cover/Interior Layout Designer;* Corinne Burton, *M.A. Ed., Publisher*

Shell Education

5301 Oceanus Drive

Huntington Beach, CA 92649-1030

http://www.shelleducation.com

ISBN 978-1-4258-0788-7

Reprinted 2013

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CHAPTER 1

The 5 Ws and an H of Problem Solving

"Problem solving develops the belief in students that they are capable of doing mathematics and that mathematics makes sense."

Van de Walle et al. 2009

Stop and Think

Respond to the questions below in your problem-solving journal.

- Think back to your own experience as a student. What do you recall about solving problems?
- What is your definition of a good problem?

What Is Problem Solving?

Do you recall "doing story problems" in mathematics? They usually applied the computation and skill practice from the text book. There was little critical thinking involved to determine what you were supposed to do with the numbers contained in the problem. Not much has changed in today's mathematics textbooks. For example, after the lesson on multiplication, there might be 15 multiplication exercises for practice, followed by three or four word problems, at least one of which is an application of multiplication.

These "problems" serve the purpose of giving students practice using computational skills in a context or a situation. However, they do not help students to develop a deeper understanding of the concept behind the skill. This deeper understanding, referred to as *conceptual understanding*, involves more than applying isolated facts or procedures. Research on learning mathematics suggests that we learn new ideas in mathematics by making connections to ideas we already understand and extending the new ideas to novel situations. This is the development of conceptual understanding. For example, what is really happening when you multiply 9 by 28? You demonstrate procedural knowledge when you can go through the steps of multiplying and reaching a product of 216. Conceptual knowledge involves understanding that 9 × 28 means you have "nine groups with 28 items in each group." You can demonstrate conceptual knowledge using words, pictures, models, or numbers. When students repeat the procedure demonstrated by the

teacher, they develop the skill of multiplication. When students develop an understanding of what multiplication means *beyond* the procedure, they can apply that understanding to a variety of situations that call for multiplication.

Routine problems, or traditional "story problems" involve situations in which the learner knows how to solve the problem based on past experiences. In solving routine problems, the learner reproduces and applies a known procedure. Problem solved!

Non-routine problems and rich mathematical tasks involve situations in which the learner does not immediately know how to reach a solution. The learner must draw upon his or her comprehension of a variety of mathematical concepts and then select and extend that understanding to develop an approach that leads to a reasonable solution. The level of rigor in non-routine problems is much greater than in traditional problems. Students must grapple with the mathematics they know, extending and applying it in a new context.

Principles and Standards for School Mathematics (NCTM 2000) defines problem solving as "engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will develop new mathematical understandings. Solving problems is both a goal of learning mathematics and a major means of doing so. Students need frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking."

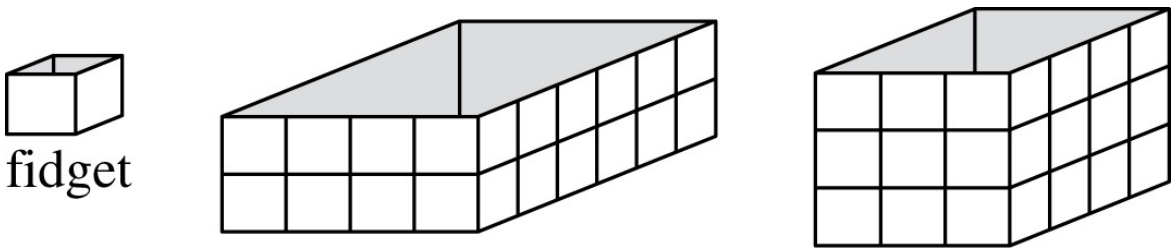
Here are examples of a routine and a non-routine problem.

Example 1

Marsha wants to fill a rectangular box with centimeter cubes. The box is 8 cm high, 4 cm wide and 6 cm deep. How many cubes will fit in the box?

Example 2

You have taken a job at the "We Box It" company, which makes cardboard boxes of different shapes and sizes for packaging and shipping products. You and your partner are in charge of boxing "fidgets," which are cube-shaped filters that are placed inside aquariums to filter the water. Fidgets are packed into larger boxes for shipping to pet stores. Here are some of the shipping boxes:



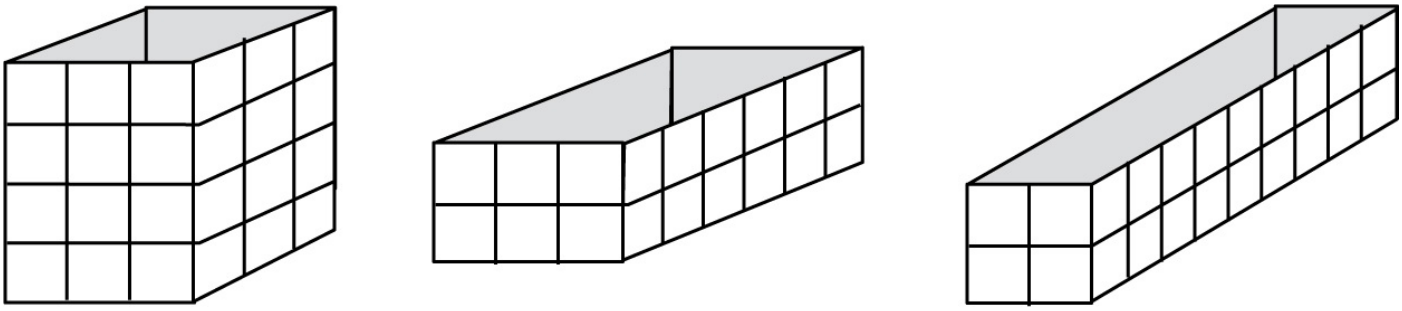
You need to pack 36 fidgets into a box. What size boxes could you use? Which box would be the best to use? Why?

Take some time to solve each of these problems. As you work on them, think about which requires procedural knowledge and which develops conceptual knowledge.

Solution to Example 1: You likely solved this problem by applying the formula, length ×

width × height (or depth) to the numbers given in the problem (8 × 4 × 6 = 192). Of course, the final solution must be labeled cubic centimeters, although many students have no idea of why that is the unit used to measure volume.

Solution to Example 2: One way to solve this problem is to count 36 cubes and build the boxes that could be used. Students may draw pictures to represent their boxes. You could organize their work in a table similar to the one shown below.



Length	Width	Height
1	1	36
1	2	18
1	3	12
1	4	9
1	6	6
2	2	9
2	3	6
3	3	4

However, the mathematics doesn't stop when the table is complete. Classroom discussion can focus on which box would be best to use—and why.

Stop and Think

Respond to the question below in your problem-solving journal.

- What mathematical concepts are developed in the fidget problem?

Example 1 is a routine problem that uses a traditional approach following a lesson on finding volume. It is expected that students will practice using the formula they learned (length × width × height) to find the volume. The task does little to build on the concept of

volume. Rather, it gives students practice in calculating the volume of a rectangular solid.

Example 2 is a non-routine or “rich” problem developing the concept of volume. In this problem students are given a situation in which they need to build an understanding of volume in order to complete the task. When solving this problem, students develop an understanding of volume by determining which three-dimensional boxes will hold 36 cubes. This is called the *volume* of the box. They discuss the various ways they found the dimensions and how they know they have all of the possibilities. Notice that this problem is likely to include a conversation about whether a box that has a length of 2, a width of 2 and a height of 9 (2 by 2 by 9) is the same as a box with the dimensions $9 \times 2 \times 2$. This idea connects to the concept of the commutative property of multiplication. The discussion will eventually conclude with students discovering that multiplying the length times the width times the height will give the volume of a rectangular prism (box).

When first faced with a problem such as the fidget problem, students may struggle because they are unfamiliar with how to approach such a task. That makes it a real problem. The role of the teacher is to be aware of the mathematical understandings developed in each problem we give our students as well as how we can support their thinking without simply showing them what to do.

In general, rich mathematical tasks or problems have the following characteristics:

- involve substantial mathematics
- have multiple points of entry
- can be solved in a variety of ways
- can have multiple solutions
- promote discussion and communication
- involve students in interpreting, testing, proving, explaining, and reflecting
- implicitly intrigue and motivate students
- can be scaffolded to meet the individual needs of students
- help to develop conceptual understanding of mathematical ideas

Stop and Think

Respond to the questions below in your problem-solving journal.

- Why is problem solving an important part of teaching and learning mathematics?
- How does problem solving support mathematical reasoning?

Why Teach Problem Solving?

The National Council of Teachers of Mathematics in *Principles and Standards for School Mathematics* (2001) recognizes problem solving to be “an integral part of all mathematics

learning, and so it should not be an isolated part of the mathematics program. Problem solving in mathematics should involve all five content areas. The contexts of the problems can vary from familiar experiences involving students' lives or the school day to applications involving the sciences or the world of work. Good problems will integrate multiple topics and will involve significant mathematics." Additionally, the other process standards—communication, connections, reasoning and proof, and representations—provide a platform for teaching problem solving. It is through these process standards that students work to make sense of and apply mathematical content.

In addition to problem solving, the process standards include communication, connections, representation, reasoning, and proof. Let's take a look at each of these processes to illustrate how they support problem solving in and out of the classroom.

Communication

The process standard of Communication involves reading, writing, talking about, and listening to others' ideas about mathematics. As students approach rich tasks and good problems, they must be able to read the problems and translate them into situations that make sense to them. They need to share ideas and strategies with others by explaining their thinking and conclusions both orally and in writing. Through class discussions, students clarify their own thinking by sharing their ideas and listening to the ideas of classmates.

Connections

The process standard for Connections includes recognizing and developing relationships among mathematical ideas. Think about the fidget problem you solved earlier. Connections between the meanings of area and volume by finding the area of the base and multiplying it by the height of the box help students use what they know to develop a strategy for finding the volume. The relationship between the commutative property and the factors of each possibility narrowed the number of possible solutions. For example, the volume of a box that is $9 \times 2 \times 2$ is the same as the volume of a box that is $2 \times 9 \times 2$. Connections are made to the units of measure used to find volume (cubic units) and why we need to measure three-dimensional space with three-dimensional units. Selecting and justifying the best box to use connects the problem to a real-world situation and provides a purpose for learning and doing mathematics.

Representation

"The ways in which mathematical ideas are represented is fundamental to how students understand and use those ideas" (NCTM 2001). There are a variety of ways students can represent their thinking as they solve problems. Pictures, words, models, tables, graphs, and numbers are among the representations used by students at all levels. Students moving from the elementary to the middle grades become more facile with symbolic representations (numbers, variables, equations) as they begin to think and reason more abstractly.

Reasoning and Proof

The process standard for Reasoning and Proof includes developing mathematical ideas, exploring phenomena, justifying results, and developing mathematical conjectures. As students solve problems, they communicate and represent their thinking, using reasoning and proof to develop and justify their thinking and solution process. They use connections to draw conclusions.

Rigorous tasks give students the opportunity to develop a deeper understanding of mathematical content through including all of the process standards. It is not a matter of choosing between content and process. Learning and doing mathematics involves both content and process. Consider the two problems involving volume that you solved earlier. Which one gives students better opportunities to reason and draw conclusions about the volume of a rectangular solid?

The NCTM Process Standards

Problem Solving

Students tackle rich tasks and problems that require them to draw on prior knowledge, employ strategies, and develop new mathematical understanding. Such tasks are thoughtfully selected, presented, and summarized so that students develop mathematical habits of mind such as persistence, curiosity, and confidence.

Communication

Students share their mathematical thinking and learn from the mathematical thinking of others; ideas are communicated in various ways, including speaking, writing, listening, drawing, and the use of manipulatives.

Connections

Students are encouraged and expected to recognize and explore the interrelatedness of mathematical ideas; the connections between mathematics and other subjects; and the relevance of mathematics to their everyday lives.

Representation

Students explore and use various ways to represent their mathematical thinking by using words, pictures, models, graphical displays, symbols including numbers and variables, expressions and equations.

Reasoning and Proof

Students make conjectures and develop ideas, including encouraging students to explain, clarify, justify, defend, and revise their thinking.

Mathematics Is Everywhere

The negative attitudes of many adults toward mathematics are likely due to the fact that they were not very successful learning mathematics. If the mathematics they did was based on drill and practice with no connection to the importance of it in our daily lives, it is no wonder that they are not excited about the possibilities mathematics offers us. We use mathematical problem solving in our daily lives. Determining the mileage my car gets

is an example of using mathematics. Looking at the finance rates and deciding whether to refinance my house involves mathematics. Knowing how much fabric to purchase for the seat cushions I want to re-cover requires mathematics. In addition to the application of mathematics in our daily lives, most careers involve mathematics as well.

In the Common Core State Standards for mathematics (2010), the standards for mathematical practice describe the expertise that all mathematics educators should develop in their students. The first of these standards is to "make sense of problems and persevere in solving them." Students need to explain the meaning of a problem to themselves and look for entry points to its solution. Providing students with experiences using a variety of strategies will help to provide that entry. Thinking about the process and ideas that are used to solve a problem are as important as the solution. Thinking should not stop when the problem is solved. Looking for other ways to solve the problem or other possible solutions often leads to new discoveries and conjectures. With this approach, the students take an active role in learning and applying mathematical ideas, and the teacher orchestrates the process.

Problem solving is a key reason for learning mathematics. It is through problem solving that we can look at a situation, analyze it, and determine possible solution paths and reasonable solutions. It is problem solving that makes mathematics meaningful in our daily lives.

Stop and Think

Respond to the question below in your problem-solving journal.

- Consider the work of an artist, a dentist, or an airplane pilot. How is mathematics used in each of those careers?

Who Should Solve Problems?

Problem solving is part of our everyday life. Consider these common daily scenarios:

- Do I take this route to work or is traffic backed up? Should I go a different way?
- Is the large box of soap a better deal than two smaller boxes?
- Do I have enough gasoline in the car to get home, or should I stop to fill up the tank?
- How do I adjust a recipe that serves 10 for my family of six?

Beyond our personal lives, we are often faced with decisions that must be influenced by our ability to reason quantitatively. What is the best site for the new community playground? How will our school district convince voters to pass a new school issue on the ballot this fall? With the advances in technology and the media, now more than ever we will all need to be informed problem solvers in order to make intelligent decisions that impact every aspect of our lives. In short, everyone needs to be able to solve problems every day.