Shell Education

Think It, Show It Mathematics

Strategies for Explaining Thinking

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Foreword by Linda Dacey

Lesley University

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Acknowledgements

With thanks to Vickie—wife, best friend,
and co-conspirator in all my projects.

Foreword

Years ago when I was teaching fourth grade, I realized that it was possible, sometimes, to feel as if I had a lens that allowed me to see into a student's brain and realize preferred representations, misconceptions, partial ideas, and clear thinking. It was a joy to discover this possibility. The same is true now that I am teaching at the college level. Gaining access to students' thinking so that I might further facilitate their learning is, for me, the essence of teaching.

When I work with teachers, I often ask them to share a classroom set of student papers with me. When I look at responses to a mathematical problem, I learn much about the teachers' expectations for mathematical thinking and communication. It is immediately clear which teachers have found some ways to support their students' ability to explain their thinking and which have not. In *Think It, Show It Mathematics: Strategies for Demonstrating Knowledge*, Gregory Denman has provided us all with the tools we need to help students make their mathematical thinking visible.

In this book, you will find step-by-step strategies for developing students' ability to write about mathematics, beginning with structured frames that scaffold learning and moving to examples of writing that invite students to elaborate in creative ways. Most importantly, the author has provided readers with mentor texts, rubrics, and student exemplars as well as ways to use student writing to inform our instructional decisions.

The *Common Core State Standards* make it clear that students must be able to understand and communicate mathematical ideas as well as gain procedural expertise. Most importantly, the Core's *Standards for Mathematical Practice* identify the processes and practices of mathematically proficient students. Briefly, these practices highlight the need for students to make sense, reason, model, use appropriate tools, attend to precision, look for and use structure, and look for and use regularity in repeated reasoning. All students are expected to develop these abilities and will be held accountable on high-stakes tests. To meet such goals, it is critical that teachers and peers have access to students' thinking. Fortunately, Gregory Denman has many strategies for us!

—Linda Dacey, Ed.D.
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Introduction

Although I was unaware of it at the time, the catalyst for the development of the activities and strategies described in this book dates back to the fall of 1999. In my home state of Colorado, that time period saw the inception of the math assessment component of the state-mandated yearly *Colorado Student Assessment Program*. Colorado public school fifth graders were tested, for the first time, in mathematics. There was much apprehension about their students' performance on the fall assessment.

When the scores were released, the results were more disappointing than had been anticipated. Fifty percent of Colorado's fifth graders had not scored as proficient. The newspaper headlines left nothing to the public's imagination. Three of the largest newspapers in the state ran these front-page proclamations the day after the results were released:

> *Half of 5th-Graders Fail Math Test*
> Rocky Mountain News, March 3, 2000
>
> *The Math Challenge: More Than Half in Test Fall Short*
> Denver Post, March 3, 2000
>
> *Half of Fifth Graders Mastering Math*
> Pueblo Chieftain, March 3, 2000

Unfortunately, that spring, the Colorado eighth graders didn't fare much better. When districts across the state started analyzing the data from the tests, they identified a persistent weakness with the written or "constructed responses" required of students. Our fifth and eighth graders demonstrated significant difficulty in communicating problem-solving procedures and their mathematical reasoning in written form.

What we saw in Colorado was not unique to students in the "Mile High State." It was mirrored across the country. Since then, more evidence of our students not only struggling with math literacy but also lagging behind other nations has been documented. The National Mathematics Advisory Panel's final report, *Foundations for Success*, cited a 2007 study that revealed that 15-year-olds in the United States ranked 25th among their peers in 30 developed nations in math literacy and problem solving.

Simply put, our students struggle with communicating their mathematical reasoning in a verbal or written form—not to mention a resistance when asked to do so. *If I got the answer correct, why do I have to explain how I got it?* is a typical response. As a result, a number of schools began to ask me to work with their students on the writing required in both the language arts and the written portions of the math test. And now, with the advent of the *Common Core State Standards*, I continue to refine the work that I do with teachers and students in order to meet those objectives. *Think It, Show It Mathematics: Strategies for Explaining Thinking*, along with its materials, is a result of my work and research with teachers and students in an effort to help all students better demonstrate their mathematical understandings through writing and discussion.

Through this work, I have found ways to merge practical and dependable writing strategies with ongoing day-to-day mathematics instruction. For example, students use "framed paragraphs" to facilitate formulas for a written explanation of the solving of a word problem and thereby learn how writing in mathematics needs to be read and sound. The language of these "mathematical framed paragraphs" later functions as mental templates that students can fall back on, as needed, as they mature and begin working with more challenging problems. By becoming proficient with these types of paragraphs, students can internalize the sound and structure of a basic math narrative.

Below is a third-grade written explanation sample. For readability purposes, students' spelling errors have been corrected throughout the book.

> To solve the problem, I first added the 9 absent boys and the 3 absent girls and found 12 students. After that, I subtracted the 12 students from the 23 students that were in class and found 11 students. Therefore, I know Alicia's class had only 11 students left.
>
> 3rd Grade Student

Another example is using the method of writing a mathematical procedural text employing "what" and "why" statements: operations and corresponding reasons. "What" statements are statements of what mathematical operations students need to do to solve the problem, and "why" statements are statements explaining why they need to use these operations with specific references to contextual details found in the stated problem. Included with these explanations are "why" words *(since* and *because)* along with a handful of good procedural transition or sequence words *(to start with, first, then, after that, second, etc.),* and the perfect wording to start a concluding sentence *(Therefore, I know):*

> In order for me to solve this problem, I first *(transition word)* multiplied 4 times $3.25 *(what statement)* because (why *word)* I needed to know how much money Jessica had earned each day for the 4 hours she babysat *(why statement).* I found she earned $13.00 a day. Since (why *word)* Jessica worked 3 days a week *(why statement),* I multiplied 3 times the $13.00 she earned each day *(what statement).* I found that equaled $39.00. Therefore, I know that Jessica had earned a total of $39.00 each week babysitting *(concluding sentence).*
>
> 5th Grade Student

Incorporated in the activity sheets students use with this strategy are "problem-solving questions" for students to address before attempting to map out the steps they need to follow in solving the problem:

- What is happening in the problem?
- What do I know?
- What is my data?
- What don't I know?

- What is the problem asking me to find out?
- What will my answer tell me?

These questions align with the goals of the *Common Core State Standards for Mathematics:* Practice Standard 1: Make Sense of Problems and Persevere in Solving Them.

> "Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt."

Common Core State Standards for Mathematics 2010, 6

Using "what" and "why statements" along with "why" words and "transition/sequence words" helps students learn to strategically construct a coherent and logical explanation of a mathematical process. If students know the math, if they can develop a solution using convincing mathematical reasoning, then they can be taught how to clearly communicate this understanding in words. The process used to teach these strategies is the basis of this book.

Mathematics teachers, like classroom teachers in all subject areas, have full, if not, filled-beyond-capacity, instructional plates. Therefore, the strategies presented in this book can easily be integrated into any classroom, with any story or word problem, and with any curriculum that teachers are using.

Working with so many students and teachers during the development of these strategies and materials has been rewarding and edifying. The desire of teachers to have practical and effective methods to help their students perform better has forced me, as well as the teachers I worked with, to refine the material that is presented in this book. Classroom teachers, mathematics specialists, and curriculum directors as well as university instructors have been continually questioned about the effectiveness of the material. They have helped tremendously by pointing out where a different wording might be used or how a certain format might be confusing to a certain age group. I am so very grateful. With their insights, the process described in this book has enabled a great many students across the country to gain confidence and proficiency when asked to explain their mathematical thinking.

—Greg

Chapter 1

Why Write in Mathematics?

The *Common Core State Standards* insist that instruction in reading, writing, speaking, and language be a shared responsibility within the school.

Let's start by posing a question I ask of teachers when conducting an in-service on the math writing process: If there weren't a test that required students to explain their mathematical thinking, would we be teaching it in our classrooms?

To answer this, we need to briefly examine the changes that have occurred in the curriculum of mathematics over the last three decades.

The last 30 years have seen a gradual but tremendous change in the instruction of mathematics. It has become much more than the traditional computation skills—adding, subtracting, dividing, working with fractions and percentages, etc. As we make our way into the challenges of today's mathematically and technologically sophisticated world, students need not only to be able to solve problems using computation but also to reason mathematically and use that same reasoning capacity in tackling real-life situations that involve mathematics. This idea is foundational in the *Common Core State Standards in Mathematics* (2010).

Much of the change and reprioritizing that we are presently seeing with the instruction of mathematics through the *Common Core State Standards* has been as a result of the original vision, work, and efforts of the *National Council of Teachers of Mathematics* (NCTM 2000). *The Standards for Mathematical Practice* with the *Common Core* describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important processes and proficiencies with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections (2000, 6).

NCTM's Five Process Standards focus the instruction of mathematics around the activities, processes, and practice with problem-solving skills—not simply finding the right answer to a specific problem. In addition to critical computational skills, students need to be able to use mathematical reasoning in determining whether their answers or solutions make sense. Students must articulate a rationale and be able to evaluate an alternative approach(es). The *Common Core State Standards for Mathematics* states that "mathematically proficient students justify their conclusions, communicate them to others, and respond to the arguments of others" (2010, 6). Explaining their solution and reasoning and defending or justifying their approach with mathematical evidence becomes an integral part of the problem-solving process. This, of course, leads the student-learner into the need for precise skills in both verbal and written communication. In the math classroom of the 21st century, students must be able to verbalize and write coherently about complex mathematical ideas in the language of mathematics.

Writing has been referred to as "thinking made visible." Indeed, the objective of the

Common Core State Standards for Mathematics is that our students learn to communicate mathematically by correctly using the appropriate symbols, vocabulary, and labeling. The standards prepare students to:

- organize and consolidate their mathematical thinking through communication.
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
- analyze and evaluate the mathematical thinking and strategies of others.
- use the language of mathematics to express mathematical ideas precisely.

Given the principles, processes, and standards articulated by the *Common Core State Standards*, the goals and implementation of state assessment tests, and the never-ending challenges of precious available instructional time that every teacher faces, let's examine the benefits of mathematical writing in the classroom for both students and their teachers.

Thinking Made Visible

This book emphasizes that writing in mathematics offers numerous benefits, as listed below:

Provides teachers with insight into

- thought processes underlying students' problem-solving skills
- students' understanding of mathematical principles in the context of real-life situations
- the academic strengths and weaknesses of their students in order to more accurately identify and align instruction and to give students specific feedback

Provides students with opportunities to

- demonstrate their competencies and understanding of mathematical principles
- recognize their own mistakes and/or faulty mathematical reasoning
- clarify and refine their thinking
- hear and evaluate alternative problem-solving strategies
- see how they can use and transfer skills from one subject area to another (language arts to math)

Here is a sample word problem that can help illustrate these benefits.

> Todd and his parents decide to turn the spare bedroom in their house into a really cool game room. It will not only be a place where Todd can play games on his computer and watch TV and movies, but it will also have a foosball table for him and his friends to enjoy. They decide they will need to carpet exactly half of the room. If the bedroom is 18 feet by 14 feet, how many square feet of carpet will Todd's parents need to buy? Explain how you found your answer.

Below and on the next page are three samples of fourth-grade written explanations.

Student Sample 1

To start with, I multiplied 14 feet by 18 feet because I need to know the area of the room. I found 252 square feet. Second, I took the 252 square feet and divided it by two because Todd's parents were only going to carpet half of the room. I found 126 square feet. Therefore, I know that they will need to buy 126 square feet of carpet.

Student Sample 2

To start with, I multiplied 14 feet by 18 feet because to find area you multiply length by width. I found 252 square feet. Next, I divided the 252 square feet by 2 because they wanted to carpet only half of the game room, and I found 126 square feet. Therefore, I know that Todd's parents needed 126 square feet of carpet for the room.

4th Grade Students

Student Sample 3

First, I multiplied 18 feet by 14 feet because I needed to find the area of the game room. I found 252 square feet. Then, I divided the 252 square feet by 2 because only half of the game room would be carpeted. I found 126. Therefore, I know that Todd's parents will need 126 square feet of carpet unless they sell the house or a natural disaster hits!

4th Grade Student

Each of these students obviously wrote a solid (and in one case clever) response. Not only did they demonstrate a conceptual understanding (difference between perimeter and area, the operation necessary to determine area, and the fact that division of a number by two equals one half of the number), but they also did each calculation correctly at the operational level. The sequenced and logical construction of their writing demonstrated successfully integrated writing skills. As suggested earlier, these samples confirm that if students know the math and use mathematical reasoning, they can be taught how to communicate that understanding with words.

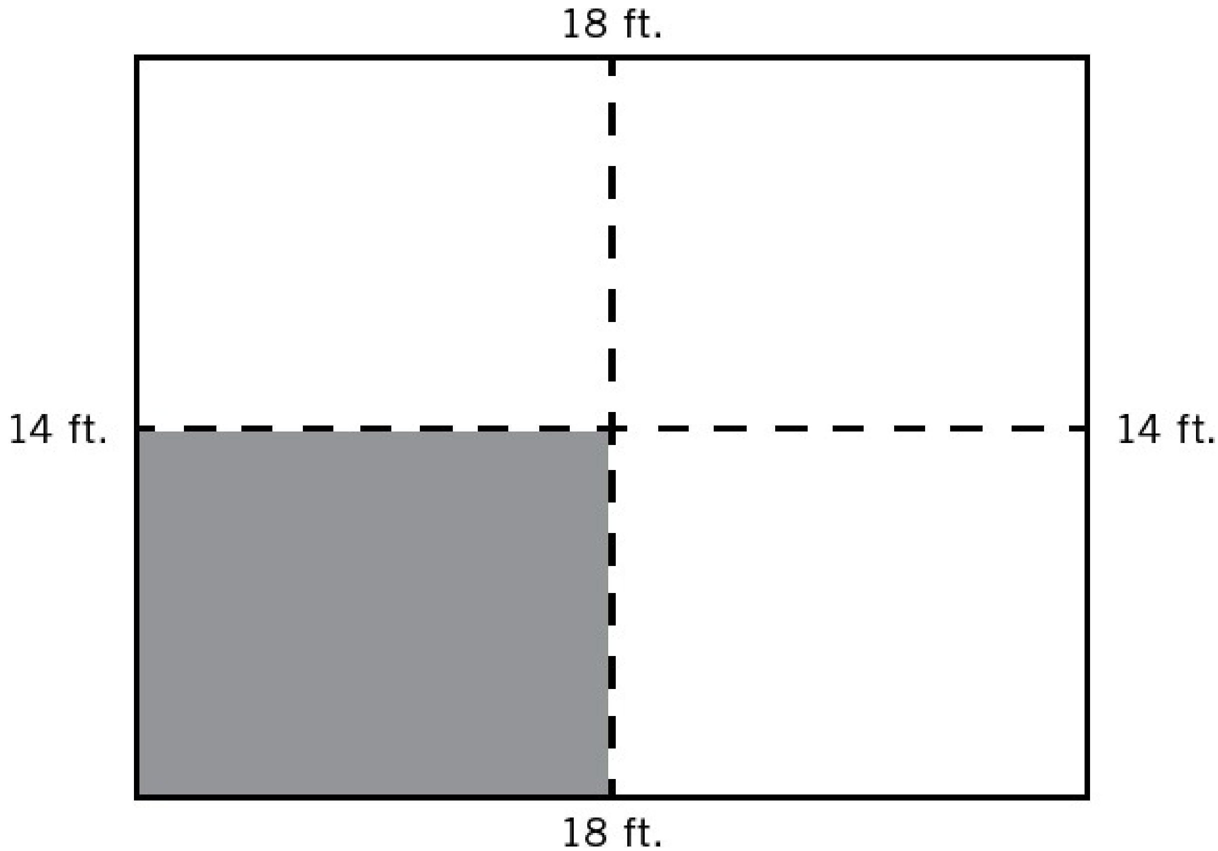
Beyond just getting a "correct" answer, it can be argued that there is power (and magic) in the pure satisfaction of a correct and logically presented mathematical explanation. By having them explain their mathematical process, students have been asked to wrestle with the construction and articulation of the thinking behind their answers, and in the examples of these students, they have done just that. It's certain that when students orally share their written responses in class, especially those who have written like the sampled students above, they read with a confidence that cries out: *I know what I am doing and how to do it!* It can be thought of as mathematically-empowered students in the making. Another student, however, addressing the same math problem did not fare so well.

Student Sample 4

Since they are only carpeting half of the room, I cut the length and width in half and then multiplied them together. I found 63 square feet. Therefore, I know that they will need 63 square feet of carpet.

4th Grade Student

At this point, the student was confused and not sure where he had gone wrong, so I simply asked him to draw the shape of the room to explain what he had done as the first step of his written response.



You could have heard the "Ah…I get it!" all the way down the hall when he discovered *on his own* that by cutting both the length and the width in half and then multiplying them, he was finding only one-fourth of the area of the room. He could now easily revise his written answer.

At the same time, another student, knowing that he was encouraged to visualize and sketch out the details of the problem the class is working on, did just that and produced an alternative solution.

Student Sample 5

To solve the problem, I first drew the 14 ft. by 18 ft. game room and decided to cut it in half down the middle. This gave me a length of 9 ft. So, I multiplied 14 ft. by 9 ft. and found an area of 126 square feet. Therefore, I know that Todd's game room will need 126 square feet of carpet.

4th Grade Student

With this student's written explanation, the entire class was able to hear and evaluate another way to solve the problem.

In order for students to progress toward mastering the skill of communicating in mathematics, they need as much specific individual feedback as possible on what they produce. Consider, for example, the work of Student 6.

Student Sample 6

First, I multiplied 14 by 18 and I found 252, and it is the area of the room. Then, I divided 2 by 252 and I found 126, and that is half of the room. Therefore, I know that is half the room.

4th Grade Student

The criteria for the nine-point rubric *Explaining Your Math Thinking in Words* is used (presented in greater detail later in the book) as a reference. See Figure 1.1.

Figure 1.1 Explaining Your Math Thinking in Words

Appendix C: Student Resources

Name: ________________________________ Date:______________

Explaining Your Math Thinking in Words

How did you do?

Rubric Criteria

Students' problem-solving explanation will be scored as follows:

		Score
Labels	• 1 Point: Limited labels are used in the explanation. • 2 Points: Correct labels are used throughout the explanation.	
Math Vocabulary	• 1 Point: Limited math vocabulary is used in the explanation. • 2 Points: Some math vocabulary is used but incorrectly in math sentences. • 3 Points: Math vocabulary is used and correctly written in math sentences.	
Problem-Solving Process Explanation	• 1 Point: The steps in the problem-solving process are incomplete or not explained; limited or no transition/sequence words are used. • 2 Points: The steps ("what") in the problem-solving process are presented but the mathematical reasoning ("why") behind each step is not; transition/sequence words are used but "why" words are not. • 3 Points: The steps ("what") in the problem-solving process are presented but not all the mathematical reasoning ("why") behind each step is presented; transition/sequence and "why" words are used but not in a logical way. • 4 Points: The steps ("what") in the problem-solving process are presented, each with the mathematical reasoning ("why") behind the steps; transition/sequence and "why" words are used correctly, but the overall explanation does not read smoothly. • 5 Points: The steps ("what") in the problem-solving process are presented, each with the mathematical reasoning ("why") behind the steps; transition/sequence and "why" words are used correctly and logically; the overall explanation is clear and reads smoothly.	

Explained the steps of the process in a way that is both logical (explained the "what" and "why" of your steps) and readable (used transition words).

Did you hit the 10? **Your Total Score:**

140 *#51051—Think It, Show It Mathematics* *© Shell Education*

Using the rubric criteria as a guide, the teacher or members of a cooperative learning group, could conference with students to give them thorough feedback to take steps toward mastering the math explanation. It should be pointed out that the student knew how to solve the problem and had calculated correctly. She used transition words *(first, then*, and *therefore)* to sequence the steps of her explanation. The student would then have to be shown how points were lost for failing to use labels *(14 what? 18 what?)*. Further, the student did not write the division operation in a correct math sentence *(2 by 252)*. We would need to write the operation as a number sentence to work on that. The conference would then conclude by having the student reread the original word problem to determine exactly what the answer should tell us *(how many square feet of carpet Todd's parents will need, not the size of half the room*), reminding the student that the concluding sentence in a math explanation must always answer exactly what the problem is asking.

Finally, by going through the entire class's written responses from the day's work, it was decided that there were four mini-lessons to be conducted that week:

- Demonstrate how including labels throughout their answers helps avoid confusion.
- Use a consistent tense (multiplied, found, needed, divided, etc.) in the written paragraph by using some examples from students' work.
- Emphasize how helpful it can be to visualize and sketch out a problem before attempting to solve it.
- Stress how important it is to always reread written answers quietly to ourselves to see if they make sense or if any words were inadvertently left out.

Writing narratives in mathematics give students the opportunity to become aware of their thinking process while problem solving. Students are able to write coherent and logical explanations of their problem-solving process. Additionally, students are afforded the opportunity to reduce or eliminate mistakes. By explaining their thought processes, students are held accountable for their work while furthering their mathematical knowledge.

Chapter 2

One- and Two-Step Problems

Problem solving in mathematics forms a foundation for student success and “is not only a goal of learning mathematics but also a major means of doing so” (NCTM 2000, 4). According to the *Common Core State Standards*, students must be able to apply mathematics to solve problems that occur in everyday life as well as explain their thinking and communicate their solution (2010). By solving mathematical problems, students experience opportunities to think mathematically, persevere in solutions, develop curiosity and confidence, and make connections to mathematics and the real world (NCTM 2000). As suggested in the opening chapter, problem solving allows teachers to build mathematically empowered students and requires students to think critically and creatively in order to arrive at solutions.

This level of thinking, problem solving, and mathematical competence is even more critical today with an increasing number of students using calculators. As essential tools in our classrooms and lives, calculators serve us by giving us right answers, but reliance on them can allow students to circumvent an understanding of the mathematical principles involved with a given situation.

I am reminded of my attempt to buy a sweater at a department store awhile back. It was a $50 sweater reduced by 20 percent. A great deal! So I picked up the sweater and headed to a counter. Unfortunately, the clerk there had misplaced her calculator and was flustered. “You’ll need to go to the other counter,” she apologized. I looked and saw a line of six or seven people. Not wanting to deal with a long wait, I simply told the clerk that 20 percent of $50.00 is $10.00 and if we subtracted that from the $50.00, the sweater would cost $40.00. “So just key $40.00 into your register and figure the tax and I’ll be on my way,” I suggested. But the clerk still insisted that she needed her calculator. “It’s not that I don’t trust you,” she confessed, “but I always use my calculator.” Finally, she had an idea and picked up her telephone, called another clerk, and explained, word-for-word, exactly what keys to press on her calculator, and they found—lo and behold—that with the 20 percent discount, the sweater would only cost $40.00.

I am not being terribly critical of the clerk (trust me—I rely heavily on my calculator when averaging grades, balancing my check book, and doing taxes!). It does, however, illustrate how important it is for students to go beyond the computations afforded by a calculator to understanding the mathematical processes and principles underlying what they key into their calculators. In the case of my department store clerk, it was to understand the function of percentages in determining the price of a discounted sweater.

Another case in point returns us to the fourth-grade students who were working with the problem of Todd and the carpeting of his game room that I discussed in the first chapter. Earlier in the year, these same students had been given activity sheets on which they practiced calculating perimeters and areas that were presented pictorially. But later in the year some of them were baffled when the two mathematical tasks were placed side-by-side in the real-life context of Todd and the square feet of carpet required for his game room. Our students need to become skilled in addressing and understanding word problems as they might be encountered in everyday life.

Addressing Word Problems

We start teaching students how to explain their mathematical thinking by showing them how to first systematically address word problems. The process follows these steps:

Read → Decide → Estimate → Work → Explain

1. **Read the Problem**
   - What is happening in the problem?
   - What do I know?
   - What don't I know?
   - What is the problem asking me to find out?
2. **Decide**
   - What operation(s) will I need to do to solve the problem?
   - What strategy will I use to solve the problem?
3. **Make an Estimation**
   - What is a reasonable answer?
4. **Work the Problem**
   - Check my work.
5. **Explain My Math Thinking in Writing**

The first step with any word problem is to establish what is happening. The *Common Core State Standards for Mathematics* describes mathematically proficient students as ones who "start by explaining to themselves the meaning of a problem" (2010, 6). Students should be regularly asked to explain in their own words what is happening in a given problem. A fun yet effective way to do this is to have students reframe what is happening as if they were telling it as a story by starting with the words *once upon a time*. Here is a student example based on the following word problem:

> Farmer Arturo had 165 cows in his pasture. One night, some pranksters opened his gate, and 39 cows wandered out into his neighbor's pasture. How many cows does Farmer Arturo have left in his own pasture? Explain how you found your solution.

> Once upon a time, there was a farmer named Farmer Arturo who had 165 cows in his pasture. But one day, some guys came in and let 39 of his cows out of his pasture. Farmer Arturo was so sad! He wanted to know how many of his cows he had left.
>
> 3rd Grade Student

Students are then asked to share their "stories" with their learning partners or to share and listen to one another in cooperative learning groups. Students are also encouraged to visualize the problem's situation and make a drawing or sketch to represent the problem situation. This is important because students often want to read problems as quickly as they can and jump immediately into doing the calculation. The desired result is "I'm the first one done!" being loudly announced as their arms and pencils are launched straight up into the air. This "being-done-first" mindset, of course, is common among many students. It is true that in the early grades, many students often are able to correctly solve some problems without careful and analytical reading. However, as they progress into more difficult and multifaceted mathematics, this assuredly will not continue to be the case. Students need to develop a mathematical "habit of thought" when working with word problems by always explaining to themselves what is happening in the problem.

Fundamental in the careful reading of a word problem in order to solve it is the determination of the essential information. They are the "givens" or "what do I know?" Along with "what do I know," students need to address "what don't I know." Determining this often involves the identification of the words.

It is important to note that not all word problems have specific clue (key) words such as these:

- How many altogether?
- How many in total?
- How many in all?
- How many fewer?
- How many would be left?
- What would remain?

Also in the wording of some problems, clue words can actually be misleading. Given this, some researchers have cautioned against using them as a strategy in the analysis of a word problem (Van DeWalle, 2009). However, many core curriculums refer to clue/key words, and students are asked to identify them when they can but only as one of the many available strategies they can use in understanding what the problem is asking. Students need to carefully read the problem in order to make sense of it. A final key element in this step is for students to circle the key number facts in the problem (i.e., *165 cows and 39 cows)* and underline the problem being asked (i.e., *How many cows does Farmer Arturo have left in his own pasture?*)

Step two asks students to determine what strategy or mathematical operation is needed to solve the problem. Some of these strategies include:

- draw a picture
- work backwards
- create a table
- find a pattern
- create a diagram
- create a list
- guess and check
- write an equation/number sentence
- use simpler numbers

Many times, drawing a picture or diagram, making a list, working a simpler version of the problem, or using patterns is the perfect strategy. As students continue working and growing in mathematics, they will become more comfortable in finding a workable way to solve a problem, or as articulated in the *Common Core State Standards for Mathematics*, "plan a solution pathway" (2010, 6). Students will discover that there may be more than one "pathway" that will lead to a correct solution. They should, therefore, start at an early age identifying and stating the strategies or pathways necessary to solve their problems. By explicitly employing problem-solving strategies and writing and discussing those strategies, students make clear the process they are following and provide teachers a window into their thinking. This is important as students progress into more difficult mathematics in order to know whether students are truly gaining conceptual understanding or simply have a rote procedural understanding of the problem (Gojak 2011).

Step three involves making an estimation to find a reasonable answer to check the final solution against. This is an important step in the process in order to help students gauge whether they are on the right track once they have completed the problem. If their estimation is way off from their final answer, students should go back and check their work in order to see if they made a calculation or other mathematical error. This is especially important when students are using calculators. Students often think of calculators as always displaying correct answers. And although a calculator will always compute correctly, students need to remember that they could have made a mistake entering the numbers or the operations, and they should check their final answer against their estimation. For example, although the calculator may display 42, that answer is unreasonable as the average points scored by a team were 25, 17, 34, and 43 points.

A simple way to work with estimations is to explain that when performing calculations, estimation is substituting numbers that we can easily add in our heads, or at least with minimal effort using paper and pencil, to give us a reasonable answer. Very often, this involves rounding and thinking in sets of easy-to-calculate numbers such as fives, tens, and hundreds. Another way to have students think about calculation estimations is to refer to them as "pre-answers."

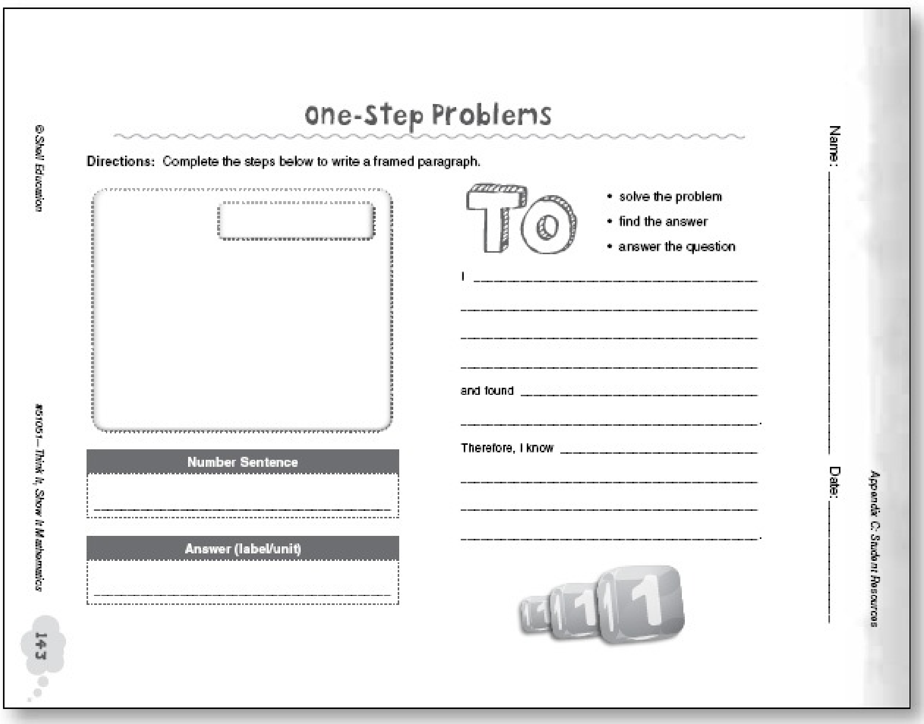
In the broader picture, the ongoing practice of making estimations allows us to circumvent possible headaches in our day-to-day lives. For example, if I am picking up fencing material to build a dog run in my backyard and know that the area measures 9 feet by 13 feet, I would first round 9 to 10 and 13 to 15. That way I can go to the hardware store having mentally calculated that 10 and 10 equals 20, and that 15 and 15 equals 30, and finally that 20 plus 30 makes 50. Therefore, 50 feet of fence is certainly going to work. Knowing this, I would not have the hassle of having to return to the store a couple of hours later to purchase another 10-foot roll because I arbitrarily guessed 40 feet would be enough. Or if I am having my son run to the grocery store to pick up three items, I can estimate how much they will cost and give him the right amount of money so he doesn't have to come back complaining that he didn't have enough and grumbling about having to make a second trip back to the store.

At this point, let's turn our attention to the last two steps of the process: working the actual problem and then explaining their problem-solving steps in words. Here, students use a "framed paragraph" in order to support the language necessary to draw a conclusion about the mathematical processes used to find and clearly communicate the solution.

One-Step Word Problems

One-step word problems involve scenarios in which only one calculation is needed in order to arrive at the solution. The *One-Step Problems* activity sheet pictured in Figure 2.1 should be used when students are working with single-step math problems. In order to be truly successful with this process, students need a framework to support the problem-solving steps and the language needed to clearly communicate their solution and conclusions. Not only is it important to provide guidance for students to frame the mathematics but also by using these framed paragraphs, students hear how writing in math "sounds." Through repeated exposure to writing and hearing these math narratives, students internalize their linguistic pattern.

Figure 2.1 One-Step Problems



Students start on the left side of the sheet with the large blank work area. They are to record the label or unit from the problem at the top of the space in the rectangle provided (e.g., puppies, basketballs, or cows). They then work the problem they identified in step two using whatever strategy they have chosen to solve the problem.

Try It!

> Some teachers have enlarged the *One-Step Problems* activity sheet included with this book and laminated it so that students can draw using an overhead pen or perhaps work with manipulatives or number strips in the actual work area on the sheet and then complete it.

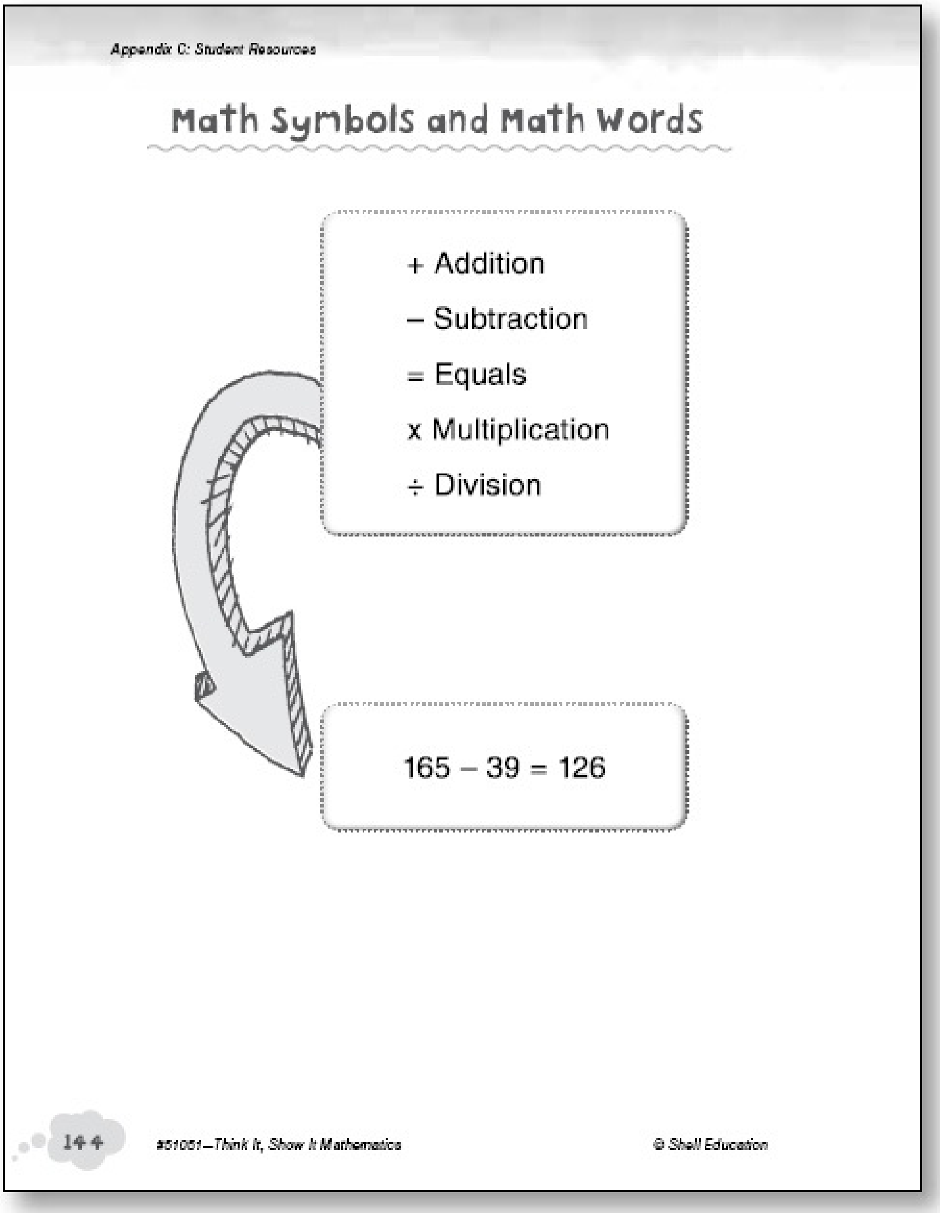
The next step is for them to write a number sentence in the correct box for the mathematical operation that they performed. This is completed during step 4 of the problem-solving process previously discussed. A correctly constructed number sentence will help them express it correctly with words in their subsequent written paragraph. For example, if they correctly write the number sentence as *165 – 39 = 126*, they more likely will write out *165 minus 39 equals 126* as opposed to reversing the minuend and subtrahend *(39 minus 165*).

Writing number sentences correctly, however, is predicated on two pieces of critical mathematical knowledge (Figure 2.2):

- Conceptual knowledge of what the mathematical operation means
- Symbolic knowledge of what math symbols (=, +, –, ×, ÷) represent in the expression of a number sentence

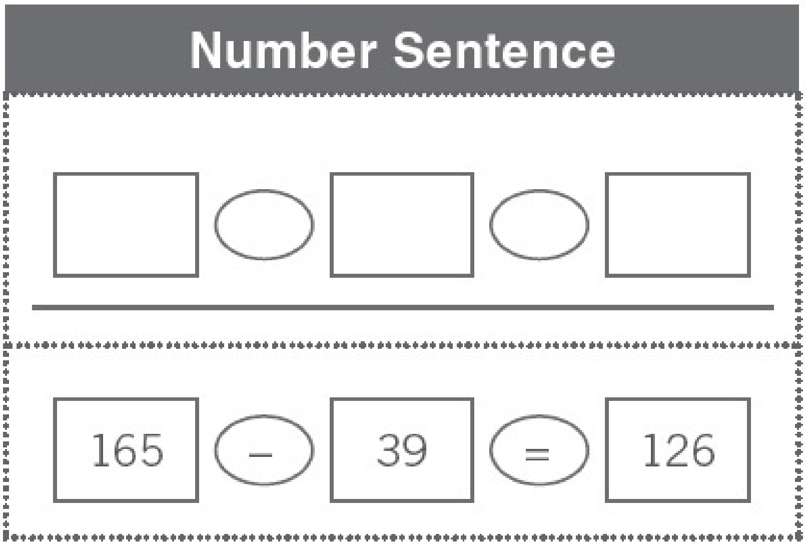
Working at the conceptual knowledge, it is generally helpful to go back to “models” of subtraction *(using drawings, physical manipulatives, number strips or number patterns, etc.)* to help the student construct an understanding of what subtraction is.

Figure 2.2 Math Symbols and Math Words



For struggling students, some teachers have found it helpful to scaffold their learning by drawing squares and circles on the Number Sentence rectangle on the *One-Step Problems* activity sheet. These guide students in writing their number sentences. Students know that numbers are written in the squares and symbols in the circles. See Figure 2.3 for a visual reference of this guide.

Figure 2.3 Number Sentence Scaffold



Finally, from the correctly constructed number sentence, students write the answer to the problem with its label or unit in the Answer (label/unit) box. This is also completed during step 4 of the problem-solving process previously discussed. Students are now ready to move to the right side of the sheet for the written explanations of their mathematical work. This is part of step 5 in the problem-solving process. Here, students complete the framed paragraph on the sheet by first selecting and circling one of the phrases to continue the sentence: To ___________

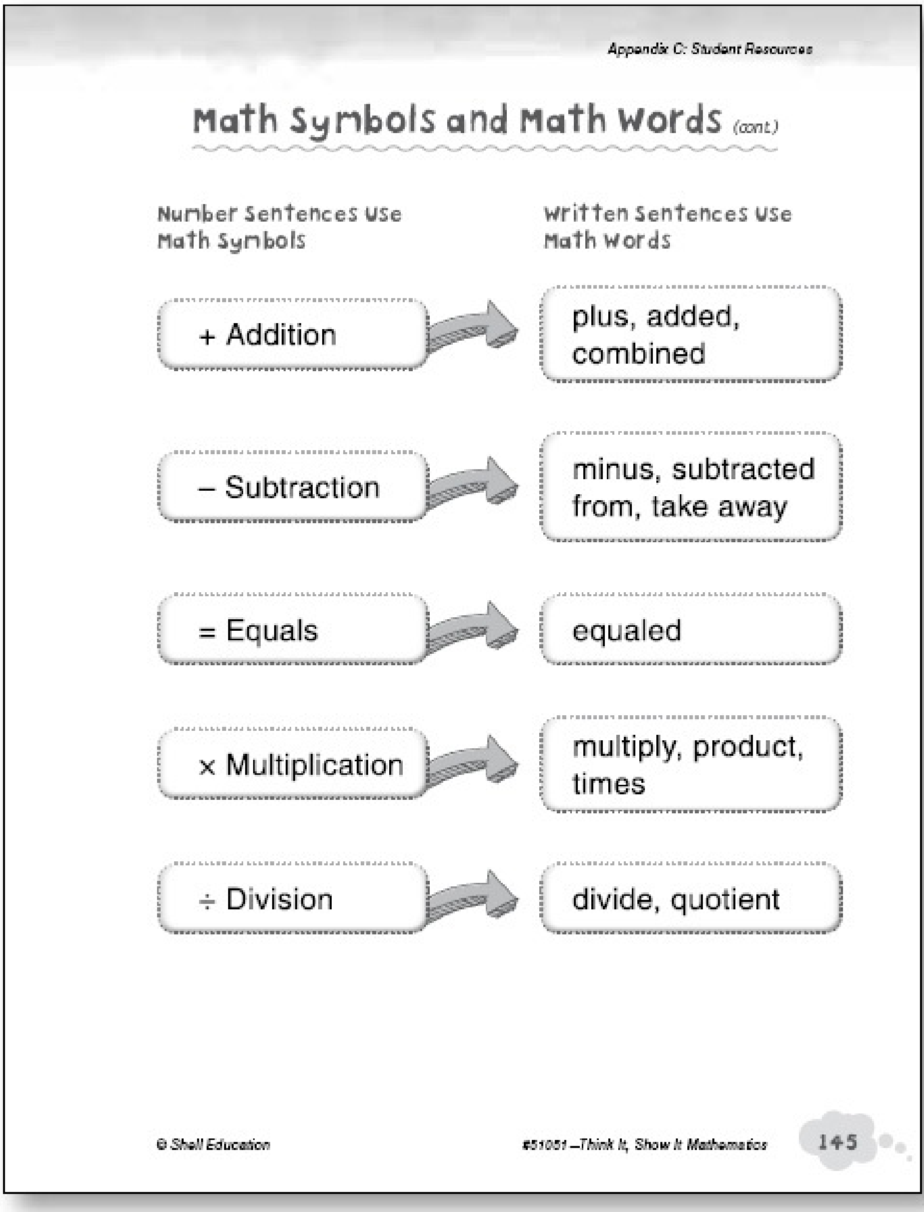
- solve the problem
- find the answer
- answer the question

All of these phrases will work perfectly in any math narrative, but by choosing, using, and hearing different ones, students not only learn a variety of ways to begin their paragraphs but also have the satisfaction of choosing phrases on their own.

Then, referring to the left side of the sheet where they have written their number sentences, they write the numbers in words, and operations they performed to arrive at their solution. Here they can refer to the chart shown in Figure 2.4 to help them determine which words to use in their sentences as well as their correct spellings. They see that when we subtract, we use the math symbol for minus (–) in our number sentences but use the written words *(minus, subtracted, from*, or *take away)* when we write our explanation in words. This sentence stresses how important it is to include the labels or units not only with their final sentence but also throughout the written responses *(I subtracted 39 what? from 165 what? and found 126 what?).* It is important for students to understand that by including labels throughout the paragraphs, they demonstrate that they understand the situation presented in the original word problem.

The final sentence starts simply with "Therefore, I know" to conclude the paragraph. Students must understand that their last or concluding sentence has to directly answer the question that they underlined in the word problem. This is further detailed on page 28. They finish their sentences, of course, by writing the solution they found in the Answer (label/unit) box. The chart shown in Figure 2.4 is helpful to display around the room at the math learning centers. Students should also keep copies of it in their math journals.

Figure 2.4 Math Symbols and Math Words Chart



Let's return to the word problem about Farmer Arturo to further investigate this four-step problem-solving process. Because the mathematics in this example is relatively simple, it is a great way to introduce the problem-solving process so that students can gain a level of proficiency with it before working with more challenging mathematical situations.

> Farmer Arturo had 165 cows in his pasture. One night, some pranksters opened his gate, and 39 cows wandered out into his neighbor's pasture. How many cows does Farmer Arturo have left in his own pasture? Explain how you found your solution.

1. Read the Problem

- What is happening in the problem?
- What do I know?
- What don't I know?
- What is the problem asking me to find out?

In order to help students understand the problem, it is important to work together to answer each question in step one.

- What is happening in the problem?
  Farmer Arturo's cows have been let out of his pasture. Some are still inside his pasture, but some are in his neighbor's pasture.
- What do I know?
  Farmer Arturo started with 165 cows. There were 39 cows let out of his pasture.
- What don't I know?
  We don't know how many cows Farmer Arturo has left in his pasture.
- What is the problem asking us to find out?
  We need to find out how many cows Farmer Arturo had left in his pasture after his gate had been left open and some of his cows wandered out.

The last two questions may appear to be repetitive and essentially contain the same information, but by having students systematically address each of them, we help them refine and practice their analytical thinking and problem-solving skills. We are not solely looking for a correct figure *(126 cows)*. We are looking for a correct answer in the specific context of the word problem's situation *(126 cows are left in Farmer Arturo's pasture)*. This level of contextual detail helps prepare students for the more advanced problems they will encounter as they progress into the higher grades. To reinforce this, it is helpful to have students underline the question asked in each of their word problems. In this case, students would have underlined as follows: *How many cows does Farmer Arturo have left in his own pasture?*

Depending on the age of students and the level of detail desired in the math problems, an alternative version of the Farmer Arturo problem can be presented. This version provides some extra information and distractors.

Farmer Arturo is a good farmer. He and his wife live in a small farmhouse with their three children. Their pasture is lined with beautiful cottonwood trees. In his pasture, Farmer Arturo has 165 cows. He takes good care of his cows. But one day, some pranksters from the town decided to play a trick on Farmer Arturo. While he and his wife were asleep, they quietly crept up to his pasture and opened his gate. "Boy," they thought, "is Farmer Arturo in for a surprise when he gets up in the morning!" And they were right. When Farmer Arturo and his family got up the next morning, they discovered that 39 of his cows had wandered out of his pasture and into his neighbor's pasture.

How many cows does Farmer Arturo have left in his own pasture?

Discussion about this problem is very similar to what was shared regarding the simpler version; however, there should be richer discourse around what is needed and not needed in order to solve this problem.

- Important information *needed* in solving the problem:
  Farmer Arturo had 165 cows in his pasture.
  He discovered that 39 of his cows wandered into his neighbor's pasture.
- Interesting but unimportant information not needed to solve the problem:
  Farmer Arturo is a good farmer.
  He takes good care of his cows.
  He has a wife and three children.
  They live in a small farmhouse.
  His pasture is lined with cottonwood trees.

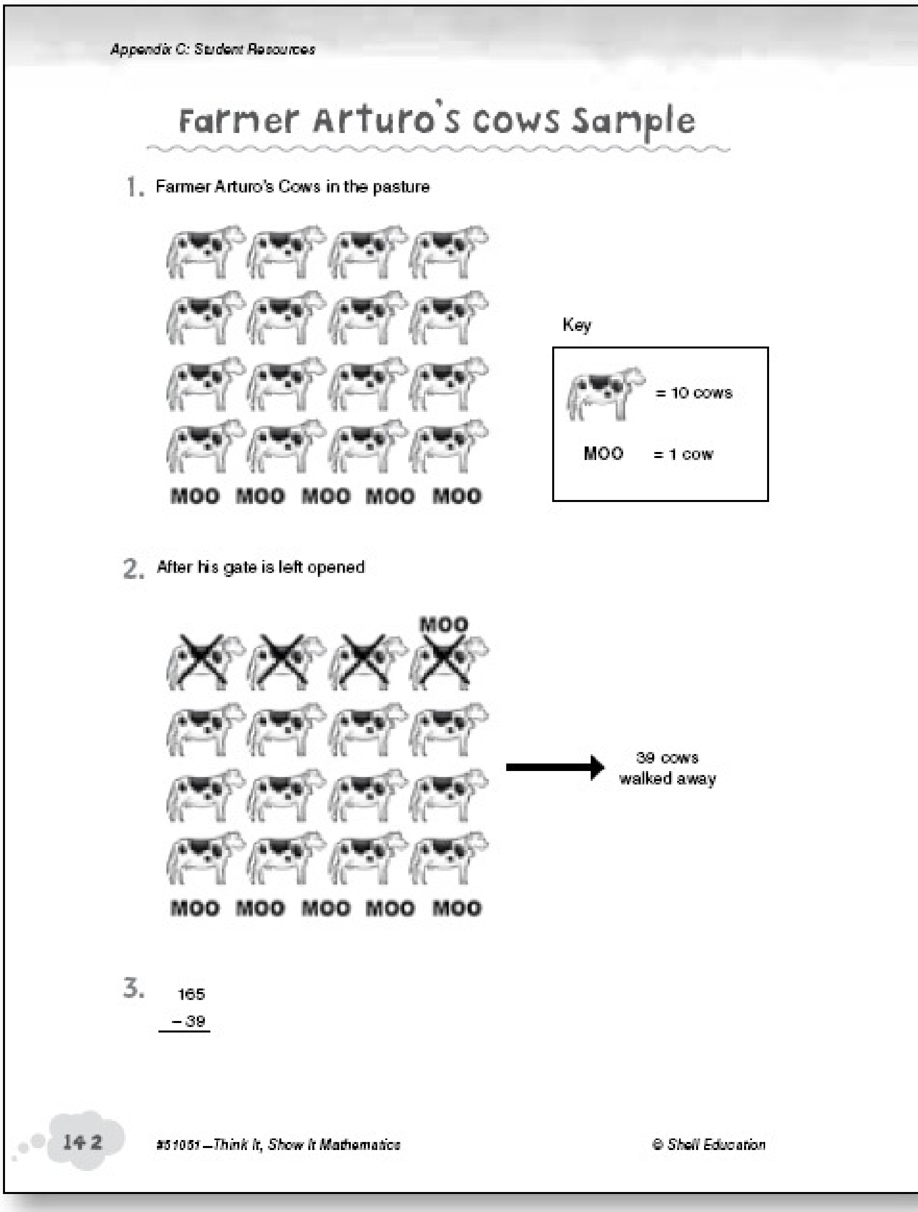
These discussions prompt students to realize that not all of the numbers presented in the problem are important to the actual solution. The fact that Farmer Arturo and his wife have three children has nothing to do with determining how many cows he has left in his pasture. In order to help students ignore the unneeded numbers, have students draw a circle around the important information necessary to solve the problem. In this problem, students would draw circles around "165 cows" and "39 cows." We are now prepared for the next step.

2. Decide

- What operation(s) will I need to do to solve the problem?
- What strategy will I use to solve the problem?

There are many problem-solving strategies that can be used to help students solve this problem. One good visual strategy for this scenario is to draw a picture. At this point, the teacher should then work at the board or on a chart with the class, constructing a visual representation of what the process would look like (Figure 2.5).

Figure 2.5 Farmer Arturo's Cows Sample



The teacher models the creation of this drawing/diagram using the think-aloud below:

> *"Okay, students, Farmer Arturo has 165 cows in his pasture. We can draw the pasture, but do you think we can actually draw 165 cows in it? Any ideas about how we can show the 165 cows? Making a key, Cindy? That's a great idea. How about using a little cow as our symbol? How many cows should each of our little cows represent? Good, Joseph, 10…that's a good strategy since we know that we can easily skip-count by tens. Now count with me and tell me how many little cows we'll need. We'll need 16 little cows, Carlos? But that only makes 160 cows. How can we show 5 more cows to make the 165? We could write the word* moo… *another wonderful idea, thank you, Liz.*
>
> *Now we need to show that 39 cows left the pasture. The number 39 is very close to 40. If we cross out 40 cows, how many cows would we need to put back into the pasture to make sure that only 39 total are subtracted? One. That is absolutely correct, class. We can show that by writing the word* moo *one*

more time. Now our pasture shows how many are left."

3. Make an Estimation

- What is a reasonable answer?

In order to make an estimation for the solution to this problem, students only need to round one number (39) in order to have a simple calculation to work with. The think-aloud previously modeled shows the beginning of using estimation with this problem. In their heads, students should be able to calculate that 165 minus 40 equals 125 and know that the final solution should be about 125 cows.

4. Work the Problem

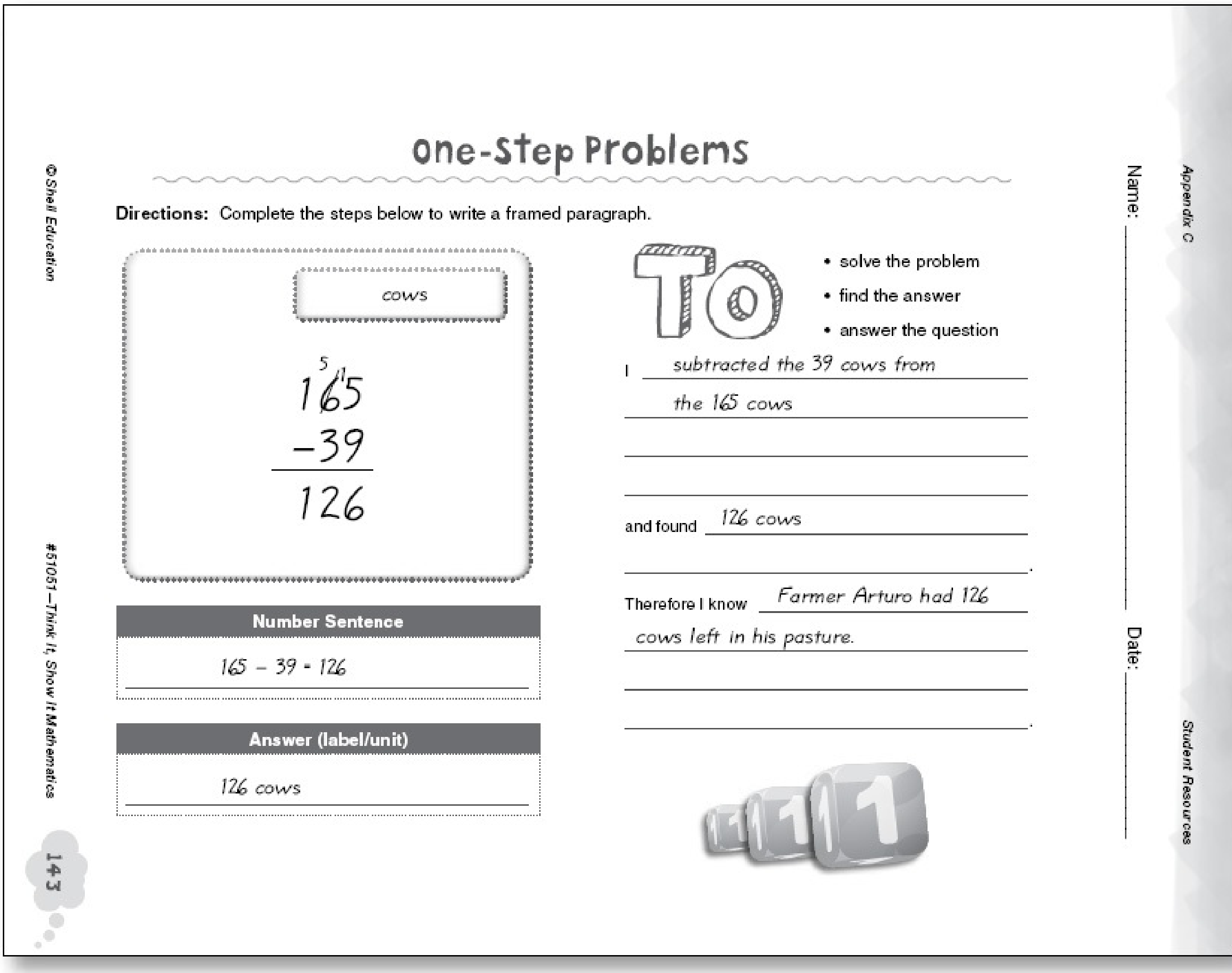
- Check my work.

5. Explain My Math Thinking in Writing

In order to work the problem, the teacher brings students back to the original drawing. By counting the remaining cows left in the picture, students can see that there are 126 cows left in the pasture. A subtraction sentence should also be used to shows this. In order to model what was completed in the picture, students should write the sentence 165 – 40 = 125 + 1 = 126. This shows that they first subtracted 40 cows and then added one cow back to make sure that only a total of 39 were actually taken away from the original 165 cows. They then record their final answer as *126 cows.*

At this point, students begin writing their paragraphs to explain their mathematical thinking using the scaffold provided in the *One-Step Problems* activity sheet. A student example of a completed *One-Step Problems* activity sheet (Figure 2.6) is located on the next page. Underneath is the student's final written paragraph. As an option, students rewrite their paragraphs in their math journals or on writing paper to be displayed on the classroom walls. It is important to look for many opportunities for students to read their paragraphs aloud and to hear others read theirs in order to practice verbalizing their thoughts and reinforce the precise mathematical language needed to explain the content and problem-solving process being used.

Figure 2.6 One-Step Problems Student Sample



One-Step Problems

Directions: Complete the steps below to write a framed paragraph.

cows

$$\begin{array}{r} 165 \\ -39 \\ \hline 126 \end{array}$$

Number Sentence

165 – 39 = 126

Answer (label/unit)

126 cows

TO
- solve the problem
- find the answer
- answer the question

I subtracted the 39 cows from the 165 cows

and found 126 cows.

Therefore I know Farmer Arturo had 126 cows left in his pasture.

Name: ____ Date: ____

Appendix C

Student Resources

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To solve the problem, I subtracted the 39 cows from the 165 cows and found 126 cows. Therefore, I know Farmer Arturo had 126 cows left in his pasture.

3rd Grade Student

Try It!

After using the *One-Step Problems* activity sheet you may want to extend your students' learning by having them take their number sentence (165 – 39 = 126) and write and illustrate their own word problems.

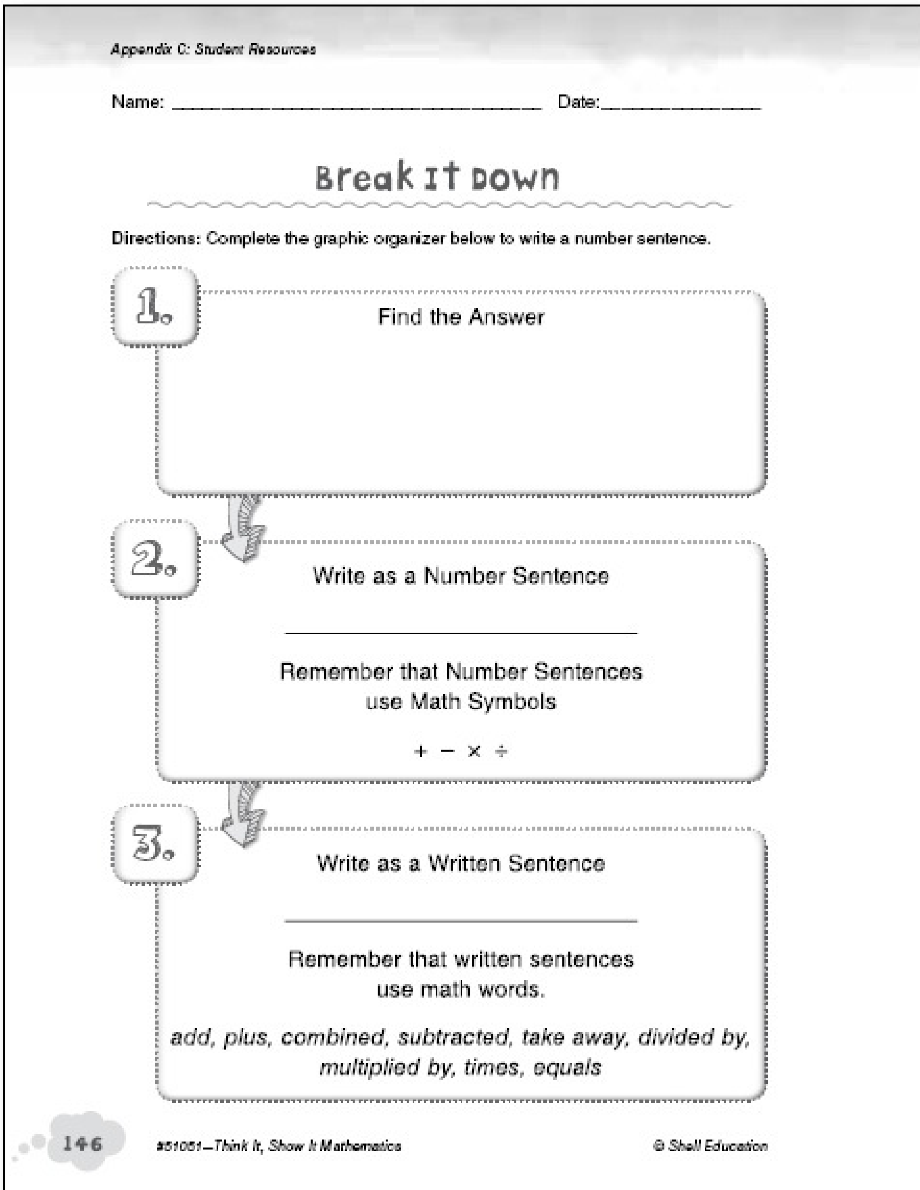
There are 165 second graders at Emerson Elementary School, but one day 39 of them were sick and didn't come to school. How many students were left at school that day?

3rd Grade Student

There may be some students who are not ready to work independently with the *One-Step Problems* activity sheet. They may struggle not only with the computation, but also with writing the number sentences and then transferring them into a written sentence.

Figure 2.7 displays an activity sheet that can be used to give these students practice with this sequence. To use this activity sheet, students must first be provided a problem. The problem is to be worked in the first box using drawings, number strips, or manipulatives—whatever is most helpful for students—to find the answer. In box two, students record the operation as a number sentence. Finally, students take their number sentence and write it as a written sentence in box three, where they have the spelling of the words they need.

Figure 2.7 Break It Down


Appendix C: Student Resources

Name: ______________________ Date: __________

Break It Down

Directions: Complete the graphic organizer below to write a number sentence.

1. Find the Answer

2. Write as a Number Sentence

____

Remember that Number Sentences use Math Symbols

+ − × ÷

3. Write as a Written Sentence

____

Remember that written sentences use math words.

add, plus, combined, subtracted, take away, divided by, multiplied by, times, equals

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Here is another example of a one-step problem solved by using the *One-Step Problems* activity sheet.

Chris wants to get cool new headbands for his three buddies to use when they play basketball at recess. He knows that each headband will cost $4.36, including tax. How much money will Chris need to buy the headbands?

> To solve the problem, $4.36 plus $4.36 plus $4.36 were added, and a total of $13.08 was found. Therefore, I know that Chris will need $13.08 to buy headbands for his three buddies.
>
> 3rd Grade Student

Between the word problem in the box and a student's written explanation just below it, much mathematical learning and demonstration of this learning has occurred.

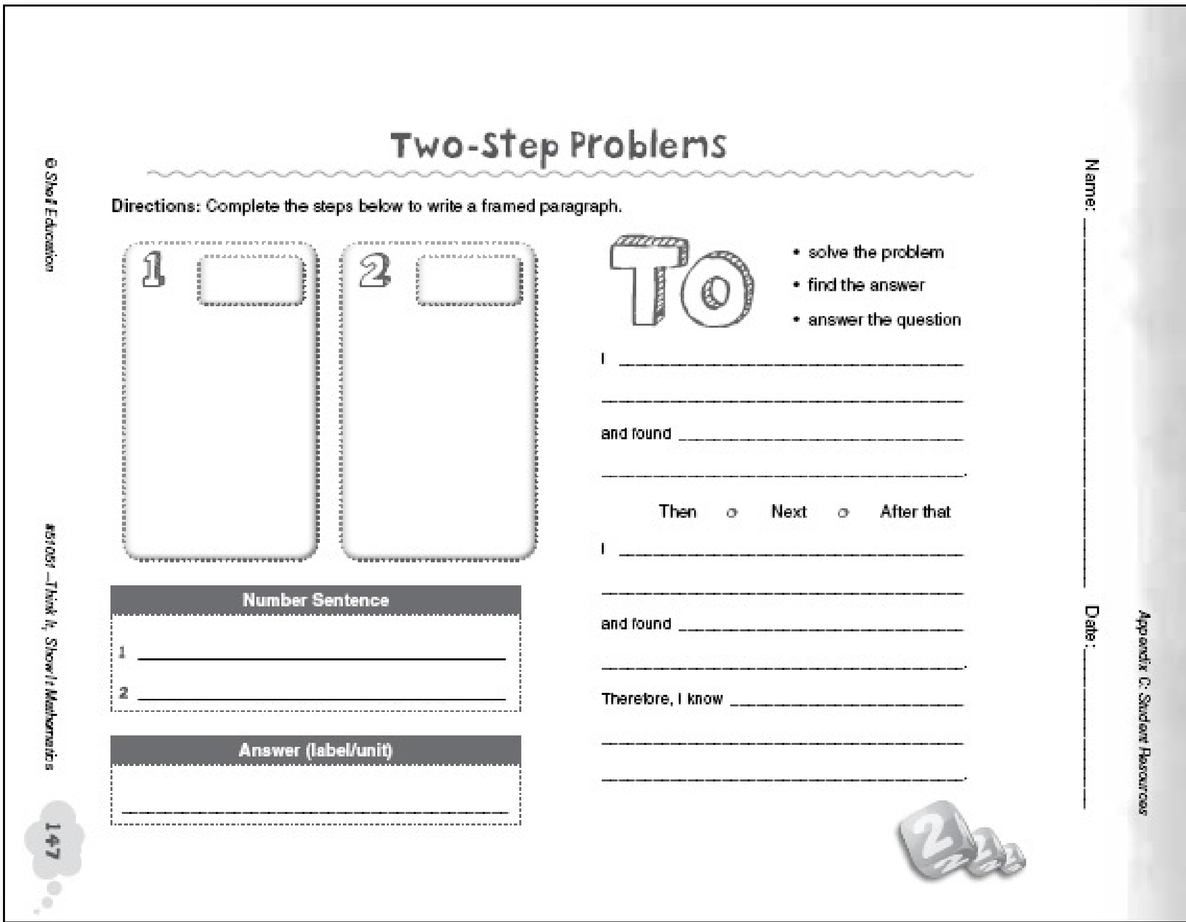
First, of course, there is the correct computational skill of adding money (*$4.36 + $4.36 + $4.36 = $13.08*), with all that it entails *(addition facts, lining up the decimals, regrouping from the hundredths to the tenths, etc.)*. Also there is the use of correct mathematical language *(using the word* [plus] *as opposed to inserting its symbol* [+]). But even before the student started the calculations, there was the actual circumstance—the context in which the problem was presented. By embedding the mathematical problem within the situation of a boy named Chris wanting to get headbands for his friends, the operation of addition was placed in a real-life scenario. The addition of money was no longer only an abstract task to be practiced and mastered, but in a broader sense, it was one of the mathematical operations that real people rely on to more easily function in the real world.

Furthermore, by placing the addition in the context of a realistic situation (as opposed to a drill or practice sheet), it allows for the classroom examination of alternative solutions. For example, the problem could be solved using multiplication ($4.36 × 3 = $13.08). In the course of further conjecture and discussion, the problem could be used to extend students' mathematical problem-solving strategies if they considered the following: What if the headbands have gone on sale at half price? How about a "buy one, get one free sale"? If Chris has $15.00, would he have enough to buy the headbands? If so, how much will he have left after he buys them?

Two-Step Problems

Two-step problems involve scenarios in which two calculations are needed in order to be able to arrive at the solution. Students follow the same problem-solving process as previously discussed—read, decide, estimate, work, explain—but multiple calculations will be completed before a solution is reached. The *Two-Step Problems* activity sheet (Figure 2.8) provides a framework for students to use while working these types of problems.

Figure 2.8 Two-Step Problems


Two-Step Problems

Directions: Complete the steps below to write a framed paragraph.

Name: ________________ Date: ________________

1

2

To
- solve the problem
- find the answer
- answer the question

I ________________________________

____

and found ________________________

________________________________.

Then ○ Next ○ After that

I ________________________________

____

and found ________________________

________________________________.

Therefore, I know ________________

____

________________________________.

Number Sentence
1 ____________________
2 ____________________

Answer (label/unit)
____

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Appendix C: Student Resources

To use the *Two-Step Problems* activity sheet, students first write their labels or units in the rectangle at the top of the first work-area box. Then, they work their first problem in the work area on the far left and write their number sentences in the Number Sentences box on line one. They do the same thing with the second operation of the problem in the second work area and write their number sentences on line two of the Number Sentences box. They record their final answers with the labels or units in the Answer (label/unit) box. A number model version of the *One-Step Problems* and *Two-Step Problems* activity sheets can be found on the Digital Resource CD.

Students then use the framed paragraph to write about their solution. They start by choosing the phrase that they want to use in the first sentence of their paragraphs and follow with the words that explain the first mathematical operations they completed. They finish these sentences by writing the answers from their first number sentences in the space after I. They start their second sentences by choosing a transition/sequence word —*then, next, after that*—and following it with the explanation of their second operation. They write this answer after the words *and found*. They complete the paragraph with *Therefore, I know* and write the solution to the problem. As suggested with the *One-Step Problem* activity sheet, students' final or concluding sentences must directly answer the question that they underlined in the problem.

Below is an example of a two-step word problem.

Miranda and Joyce love to collect and trade seashells when they are at the beach together. One afternoon, Miranda had collected 31 shells when she and Joyce started trading. She traded 8 of her biggest shells for 15 of Joyce's small shells. How many shells did Miranda have then? Explain how you worked the problem.

Just as with one-step problems, students need practice reading, setting up, and solving these types of problems in order to be successful independently.

1. READ THE PROBLEM

- What is happening in the problem?
- What do I know?
- What don't I know
- What is the problem asking me to find out?

With step 1, students should mentally visualize the situation, explain it in their own words to a learning partner or in a small group, and/or draw a visual representation of what is happening in the problem. Also, the whole class would discuss whether the fact that Miranda traded her biggest shells for Joyce's small shells is important information when answering what the problem is asking them to figure out. As with the *One-Step Problems* activity sheet, students circle the number facts and underline the question of the problem.

As a result of careful reading and discussion, students should be able to answer the following questions:

- **What is happening in the problem?**
  Two girls, Miranda and Joyce, like to trade seashells when they are at the beach
- **What do I know?**
  Miranda collected 31 seashells.
  Miranda gave 8 of her shells to Joyce.
  Joyce gave Miranda 15 of her shells.
- **What don't I know?**
  I don't know how many seashells Miranda ended up with after they traded.
- **What is the problem asking me to find?**
  The problem is asking me to find out how many shells Miranda had at the end.

2. Decide

- What operation(s) will I need to do to solve the problem?
- What problem-solving strategy should I use?

With step 2, students should identify the operations that are needed to solve the problem. When doing this type of problem for the first time, it is important to do this as a class in order for students to see that one calculation must be completed in order for the final solution to be correct. They need to understand that order matters. In this problem, the clue words *how many…have then* are not immediately helpful in determining the needed operations, so students need to comprehend the overall situation of the problem in order to solve it. The first operation and calculation that must be completed is subtracting 8 from 31—the shells that Miranda gave Joyce. The second operation and calculation is adding 15 to the number of shells that Miranda now has—the difference found in the first problem. Discussion could lead to another way to solve the problem—adding the difference between the traded shells and Miranda's total (15 – 8 = 7, 31 + 7 = 38 shells)—so be mindful to praise alternative solutions when problems can be solved multiple ways.

There are many problem-solving strategies that could be used to solve this problem, the most obvious being to draw a picture and to use an equation. Students should have the opportunity to choose a strategy that makes sense to them. However, feedback should be given in order to help students understand over time which strategies are most efficient for particular types of problems.

3. Make an Estimation

- What is a reasonable answer?

In step 3, estimation is used to get a ballpark solution to the problem. This, again, is important so that students have a way to gauge how reasonable their final solutions are. Here is one student's process for using estimation with this problem:

> Since 31 can be rounded to 30 and 8 rounded to 10, I can subtract 10 from 30 in my head, giving me 20. Then, I add the 3 fives in 15 by skip-counting from 20 by 5s… 20, 25, 30, 35. So my estimate is 35 shells.

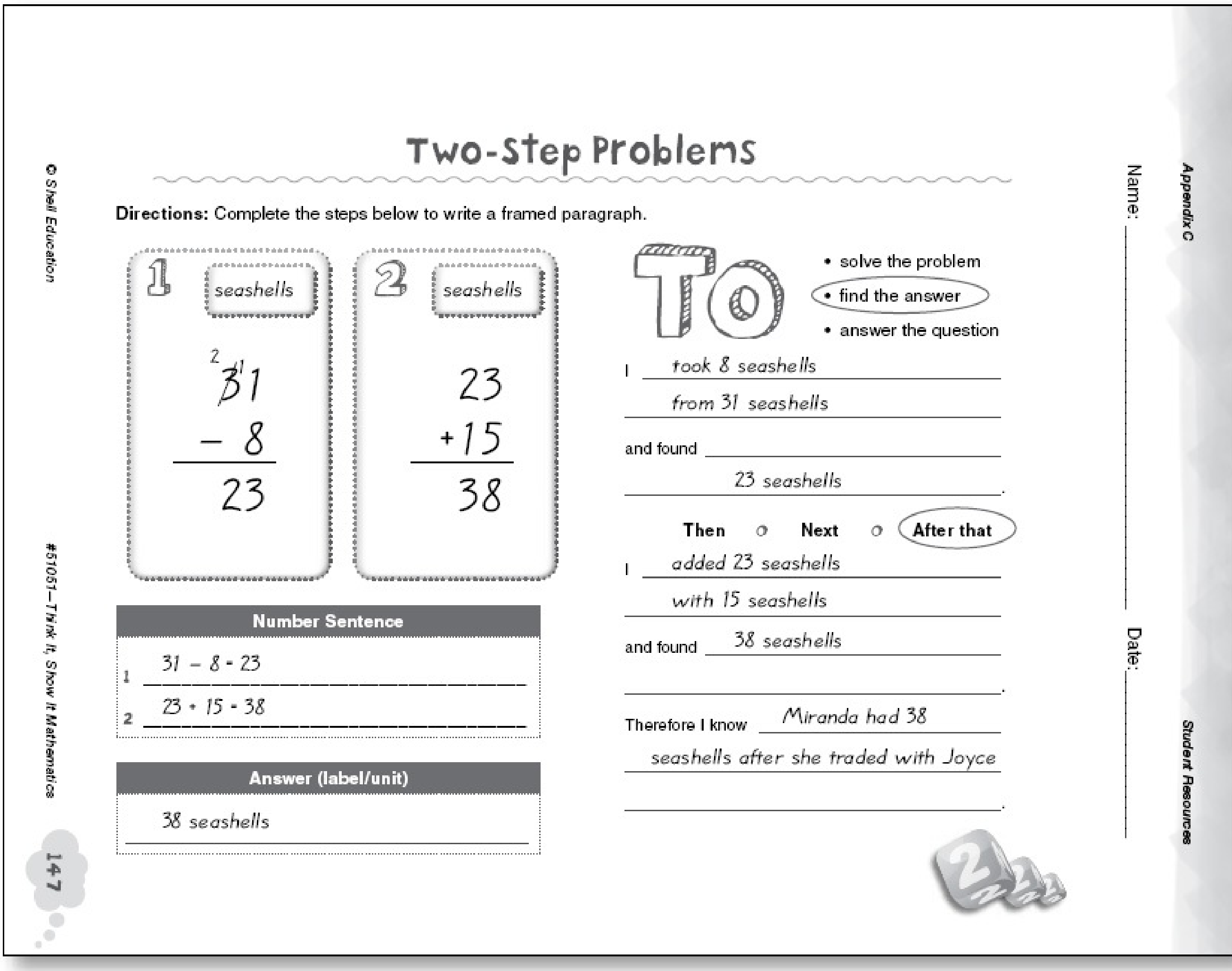
4. Work the Problem

- Check my work.

5. Explain My Math Thinking in Writing

Students complete the process by using the *Two-Step Problems* activity sheet to support them as they solve the problem and write about their mathematical thinking. On the next page is an example of a student-completed *Two-Step Problems* activity sheet and the final written paragraph (Figure 2.9).

Figure 2.9 Sample Two-Step Problems



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Appendix C

Name: ______ Date: ______

Two-Step Problems

Directions: Complete the steps below to write a framed paragraph.

1 seashells

$$\begin{array}{r} 31 \\ -\ 8 \\ \hline 23 \end{array}$$

2 seashells

$$\begin{array}{r} 23 \\ +15 \\ \hline 38 \end{array}$$

Number Sentence

1 31 − 8 = 23

2 23 + 15 = 38

Answer (label/unit)

38 seashells

To
- solve the problem
- find the answer
- answer the question

I took 8 seashells from 31 seashells and found 23 seashells.

Then ○ Next ○ After that

I added 23 seashells with 15 seashells and found 38 seashells.

Therefore I know Miranda had 38 seashells after she traded with Joyce.

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> To find the answer, I took 8 seashells from 31 seashells and found 23 seashells. After that, I added 23 seashells with 15 seashells and found 38 seashells. Therefore, I know Miranda had 38 seashells after she traded with Joyce.
>
> 3rd Grade Student

Final Thoughts on One- and Two-Step Procedural Writing

The *One-Step Problems* and *Two-Step Problems* activity sheets lay the groundwork for procedural writing in mathematics. They prompt and guide students to write clearly and precisely and to scaffold the use of the language and vocabulary of mathematics. By becoming familiar with its linguistic structure and terminology, students learn what mathematical writing "sounds" like. As they grow older and encounter more difficult problems and mathematical concepts, they'll be able to explain their problem-solving strategies with greater confidence. They are, after all, mathematically empowered students in the making.

Here are other students' examples of how Miranda and Joyce's seashell trading session might have been worked and explained. You will notice how these students expanded the language of the framed paragraph to include math vocabulary words such as *difference* and *sum* and also how some teachers want their students to spell out the number words.

> To answer the question, I subtracted the 8 seashells from 15 seashells and found a difference of 7 seashells. Next, I added 7 seashells plus 31 seashells and found a sum of 38 seashells. Therefore, I know that Miranda ended up with a total of 38 seashells to take home.
>
> 3rd Grade Student

> To solve the problem, I worked the problem of thirty-one seashells minus eight sea shells and found a difference of twenty-three. Next, I added the twenty-three seashells and fifteen seashells and found thirty-eight seashells. Therefore, I know that Miranda had a grand total of thirty-eight seashells.
>
> 3rd Grade Student

It may appear to some that these framed paragraphs produce a stilted, rather prescriptive-sounding text in contrast to the natural, more creative expression of students. It can be argued, however, that this is precisely what students are meant to do in the early stages of the learning process. If they internalize the basic language to use with a procedural explanation, then they can focus on the mathematics and simply fall back on what language to use. It becomes very natural to them. These framed paragraphs act as what I call a "mental template." Furthermore, after students have gained a level of proficiency with these paragraphs, they can be encouraged to embellish them a bit, as is shown in the final example below.

> Michael and his friends were always counting things. One day, they wanted to see how many DVDs they had between them. They asked me to help. So, first we counted how many each had. Then to find the answer, I added Michael's 6 DVDs and Sid's 12 DVDs, and Sean's whopping 54 DVDs and found 72 DVDs. Therefore, we knew that Michael and friends had 72 DVDs altogether.
>
> 3rd Grade Student

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